DYNAMICAL GROUP FOR THE QUANTUM HALL EFFECT

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Abstract

A dynamical group for the single-particle (non-interacting) Quantum Hall Effect is found, and is used to describe the Landau levels and determine the transverse (Hall) current.

1 Introduction

An area of intense interest recently, both experimentally and theoretically, has been the Quantum Hall Effect. (See, for example, Reference [1] and [2] for a review and reprint compilation on the Integral and Fractional effects.) A rigorous treatment would require a microscopic interacting many-electron approach. Nevertheless, many features of the effect may be observed in a classical treatment (the Hall current), or a single-particle quantum treatment (the Landau levels). In this note we describe a dynamical group G for the latter case. Features of this group G will remain in the interacting case, for which it must be the appropriate dynamical group in the noninteracting limit. In the absence of an electric field, the Landau levels appear naturally as the coherent states associated with G, while in the presence of an electric field E, the transverse Hall current arises naturally as an expectation in E-coherent states.

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2 The dynamical group: $E = 0$ case

The hamiltonian H_M for an electron in a magnetic field is

$$H_M = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \quad (c = 1) \tag{1}$$

which, for a constant magnetic field B along the z-axis, may be written as

$$H_M = \frac{1}{2m}(p_x - \frac{eB}{2}y)^2 + \frac{1}{2m}(p_y + \frac{eB}{2}x)^2 \tag{2}$$

or equivalently

$$H_M = \frac{1}{2m}({p_x}^2 + {p_y}^2) + \frac{1}{2}K(x^2 + y^2) + \frac{1}{2}\Omega L_z$$

$$(\Omega = eB/m,\ K = e^2B^2/4m,\ \hbar = 1)$$

where we have chosen the symmetric gauge

$$\mathbf{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0) \tag{3}$$

and the angular momentum operator L_z about the z-axis is given by

$$L_z = (xp_y - yp_x). \tag{4}$$

Introduce boson operators a_1, a_2 in the usual way [3]

$$a_1 = (\frac{1}{2\ell})x + i\ell p_x \tag{5}$$

$$a_2 = (\frac{1}{2\ell})y + i\ell p_y \tag{6}$$

$$(\ell \equiv (eB)^{-\frac{1}{2}})$$

which satisfy

$$[a_1, {a_1}^\dagger] = 1, \quad [a_2, {a_2}^\dagger] = 1, \quad [a_1, a_2] = 0. \tag{7}$$

In terms of these operators, the hamiltonian Eqn.(2) becomes

$$\begin{aligned} H_M &= \frac{1}{2}\Omega({a_1}^\dagger a_1 + \frac{1}{2}) + \frac{1}{2}\Omega({a_2}^\dagger a_2 + \frac{1}{2}) + \frac{1}{2}\Omega i(a_1 a_2^\dagger - a_2 a_1^\dagger) \\ &= \frac{1}{2}\Omega({a_1}^\dagger a_1 + {a_2}^\dagger a_2 + 1) + \frac{1}{2}\Omega L_z \end{aligned} \tag{8}$$

where the angular momentum operator L_z is

$$L_z = i(a_1 a_2^\dagger - a_2 a_1^\dagger). \tag{9}$$

The hamiltonian Eqn.(8) is an element of the Lie algebra $u(2)$ generated by $\{a_i^\dagger a_j : (i,j) = 1,2\}$, which is therefore its spectrum-generating algebra[4] (SGA) (or dynamical algebra). This is the case where there is no electric field E present. Additionally, we emphasize that in the present context we are taking a single-electron viewpoint; for a system of *non-interacting* electrons, the SGA would be a direct sum $\bigoplus u(2)$. Note that the *symmetry* group is the $SO(2)$ of rotations about the z-axis generated by L_z, since $[L_z, H_M] = 0$.

Even at this elementary level, the SGA enables a simple and immediate interpretation of the Landau levels associated with the hamiltonian Eqn.(2). Choose a conventional $u(2)$ basis

$$\{J_0, J_1, J_2, J_3\} = \{\frac{1}{2}(n_1+n_2),\ \frac{1}{2}(a_1 a_2^\dagger + a_2 a_1^\dagger),\ \frac{i}{2}(a_1 a_2^\dagger - a_2 a_1^\dagger),\ \frac{1}{2}(n_1 - n_2)\} \tag{10}$$

and for brevity define

$$A^\dagger \equiv ({a_1}^\dagger, {a_2}^\dagger). \tag{11}$$

We may represent a typical (Bogoliubov) rotation by

$$R_k^\dagger(\theta) A R_k(\theta) = \exp(i\theta\sigma_k) A \tag{12}$$

where

$$\{\sigma_1,\ \sigma_2,\ \sigma_3\} = \{\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\} \tag{13}$$

$$\{R_1(\theta),\ R_2(\theta),\ R_3(\theta)\} = \{e^{(iJ_1\theta)},\ \ e^{(iJ_2\theta)},\ \ e^{(iJ_3\theta)}\} \tag{14}$$

The energy eigenvalue equation for the hamiltonian H_M as

$$H_M \psi(n_1, n_2) = E(n_1, n_2)\psi(n_1, n_2) \tag{15}$$

and H_M is represented (apart from the $\frac{1}{2}\Omega$ constant additive term) by a 2×2 matrix M_M

$$H_M = \frac{1}{2}\Omega A^\dagger M_M A \tag{16}$$

with

$$M_M = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}. \tag{17}$$

In this basis the angular momentum operator L_z Eqn.(9) is represented by

$$M_L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{18}$$

Diagonalisation of the hamiltonian corresponds to taking a unitary (Bogoliubov) transformation on A

$$A \to U^\dagger A U = DA \tag{19}$$

(the matrix D gives a realization of the operator U) so that

$$H \to U^\dagger H U = H_M^D \tag{20}$$

is diagonal (a function only of the number operators n_1, n_2), so that

$$H_M^D |n_1, n_2\rangle = E(n_1, n_2)|n_1, n_2\rangle \tag{21}$$

with

$$\psi(n_1, n_2) = U|n_1, n_2\rangle. \tag{22}$$

Diagonalisation is effected by taking

- Case (i) $U = R_1(\pi/2)$

$$M_M \quad \to \quad \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad H_M^D = \Omega(n_1 + \frac{1}{2}) \tag{23}$$

$$M_L \quad \to \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad L^D = (n_1 - n_2) \tag{24}$$

- Case (ii) $U = R_1(-\pi/2)$

$$M_M \quad \to \quad \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \quad H_M^D = \Omega(n_2 + \frac{1}{2}) \tag{25}$$

$$M_L \quad \to \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad L^D = (n_2 - n_1) \tag{26}$$

In either case, there is a degeneracy, with only one eigenvalue labelling the energy levels (Landau levels).

The simultaneous eigenstates $\psi(n_1, n_2)$ of H_M and L_z are given by

$$
\begin{aligned}
\psi(n_1,n_2) &= R_1(\theta)|n_1,n_2\rangle \\
&= \frac{1}{\sqrt{(n_1!n_2!)}} R_1(\theta)({a_1}^{\dagger})^{n_1}({a_2}^{\dagger})^{n_2}|0,0\rangle \\
&= \frac{1}{\sqrt{(n_1!n_2!)}} R_1(\theta)({a_1}^{\dagger})^{n_1}({a_2}^{\dagger})^{n_2}R_1(\theta)^{\dagger}|0,0\rangle \\
&= \frac{1}{\sqrt{(n_1!n_2!)}} (a_1^{\dagger}+ia_2^{\dagger})^{n_1}(a_2^{\dagger}+ia_1^{\dagger})^{n_2}|0,0\rangle \qquad (27)
\end{aligned}
$$

using

$$R_1(\theta)A{R_1}^{\dagger}(\theta) = e^{-i\theta\sigma_1}A \qquad R_1(\theta) = e^{i\theta J_1} \qquad (28)$$

so that

$$a_1 \to (a_1 - ia_2)/\sqrt{2} \qquad a_2 \to (a_2 - ia_1)/\sqrt{2} \qquad (\theta = \pi/2) \qquad (29)$$

The eigenstates Eqn.(27) are $u(2)$ coherent states.

3 The dynamical group: $E \neq 0$ case

We now introduce an electric field $\mathbf{E} = (E_1,\ E_2,\ 0)$ with associated term

$$
\begin{aligned}
H_E &= exE_1 + eyE_2 \\
&= e\ E_1\ell(a_1 + a_1^{\dagger})\ +\ e\ E_2\ell(a_2 + a_2^{\dagger}) \qquad (30)
\end{aligned}
$$

The total hamiltonian H in the presence of magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ is given by

$$
\begin{aligned}
H &= \frac{1}{2m}(p_x - \frac{eB}{2}y)^2 + \frac{1}{2m}(p_y + \frac{eB}{2}x)^2 + exE_1 + eyE_2 \\
&= \frac{1}{2}\Omega({a_1}^{\dagger}a_1 + {a_2}^{\dagger}a_2 + 1) + \frac{i}{2}\Omega(a_1a_2^{\dagger} - a_2a_1^{\dagger}) + \frac{1}{2}\Omega(\mathcal{E}_1(a_1 + {a_1}^{\dagger}) \\
&\quad + \mathcal{E}_2(a_2 + {a_2}^{\dagger})) \qquad (31)
\end{aligned}
$$

in units such that $\mathcal{E}_i \equiv \frac{2e\ell}{\Omega}E_i$. The SGA in this case is the 9-dimensional algebra $\mathcal{L}$ generated by $\{a_i^{\dagger}a_j\ \ a_i^{\dagger}, a_j, I\ :\ (i,j) = 1,2\}$. The structure of

this algebra may be elucidated from a Levi-Malcev decomposition via its maximal solvable radical $\mathcal{N}$

$$\mathcal{N} = \{a_1, a_1^\dagger, a_2, a_2^\dagger, n_1 + n_2, I\} \quad (n_i \equiv a_i^\dagger a_i) \tag{32}$$

with

$$\mathcal{L}/\mathcal{N} = \{a_1 a_2^\dagger, a_2^\dagger a_1, n_1 - n_2\} = su(2) \tag{33}$$

giving $\mathcal{L} = su(2) ⓢ \mathcal{N}$. The corresponding group $\mathcal{G}$ is generated by the unitary actions on $\mathcal{L}$ of the form $y \to e^x y e^{x^\dagger}$ where x, y are (anti-hermitian) elements of $\mathcal{L}$. These unitary actions are:

- Rotations of the form

$$\begin{aligned} a_1 &\to \lambda a_1 + \mu a_2 \\ a_2 &\to -\overline{\mu} a_1 + \overline{\lambda} a_2 \quad (|\lambda|^2 + |\mu|^2 = 1), \end{aligned} \tag{34}$$

- displacements

$$\begin{aligned} a_1 &\to a_1 + \lambda_1 \\ a_2 &\to a_2 + \lambda_2 \end{aligned} \tag{35}$$

- and phase transformation

$$\begin{aligned} a_1 &\to e^{i\phi} a_1 \\ a_2 &\to e^{i\phi} a_2. \end{aligned} \tag{36}$$

This gives rise to the 8-dimensional inhomogeneous unitary group $IU(2)$ (the Center of $\mathcal{L}$ does not contribute to the unitary actions). By extending the definition of A in Eqn.(11) to

$$A^\dagger \equiv ({a_1}^\dagger, \ \ {a_2}^\dagger, \ \ 1) \tag{37}$$

we may realize the dynamical group (ignoring the phase transformation Eqn.(36) above) by

$$G = \left\{ \begin{bmatrix} \lambda & \mu & \lambda_1 \\ -\overline{\mu} & \overline{\lambda} & \lambda_2 \\ 0 & 0 & 1 \end{bmatrix} \mid \ |\lambda|^2 + |\mu|^2 = 1, \quad \lambda, \mu, \lambda_1, \lambda_2 \in \mathcal{C} \right\}. \tag{38}$$

The hamiltonian Eqn.(31) may be written in this basis as $H = \frac{1}{2}\Omega A^\dagger M A$ where

$$M = \begin{bmatrix} 1 & -i & \mathcal{E}_1 \\ i & 1 & \mathcal{E}_2 \\ \mathcal{E}_1 & \mathcal{E}_2 & 1 \end{bmatrix} \tag{39}$$

for $\mathcal{E}_i$ real.

4 The Hall current

The hermitian hamiltonian Eqn.(31) may be diagonalised by a unitary transformation. However, it is of value to obtain a canonical form under the transformations Eqn.(34) and Eqn.(35); this will enable us to obtain the coherent states of the system in the presence of an electric field, which play a role in the Hall effect. It is straightforward to prove that for singular M_H (the case here) the matrix M may not be sent to diagonal form under inner automorphisms of the algebra $\mathcal{L}$, that is, by transformations of the group G. For example, choosing the electric field $\mathbf{E}$ along the x-axis ($E_2 = 0$), under the transformation,

$$M \to T^\dagger M T \tag{40}$$

where T represents the unitary transformation $R_1(\frac{\pi}{2})D(\lambda_1, \lambda_2)$ for displacements

$$\lambda_1 = -\frac{\mathcal{E}_1}{2\sqrt{2}} \quad \lambda_2 = -i\frac{\mathcal{E}_1}{4\sqrt{2}}. \tag{41}$$

the canonical form is

$$M \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \frac{-i\mathcal{E}_1}{\sqrt{2}} \\ 0 & \frac{i\mathcal{E}_1}{\sqrt{2}} & 1 \end{bmatrix}. \tag{42}$$

This corresponds to the Bogoliubov transformation in Fock space

$$\begin{aligned} H &\to U^\dagger H U \\ &= \Omega({a_1}^\dagger a_1 + \frac{1}{2}) + \frac{i\Omega}{2\sqrt{2}}\mathcal{E}_1(a_2 - {a_2}^\dagger). \end{aligned} \tag{43}$$

$$\tag{44}$$

Although this is not diagonal in the number operators, it is simply a sum of a commuting number operator and momentum

$$H \to \Omega(n_1 + \frac{1}{2}) - \frac{\sqrt(2)E_x}{B}p_y \tag{45}$$

from which the eigenstates $|n_1, k_y\rangle$ are immediate.

We now calculate the Hall current by this method. The canonical current operator for an electron in the presence of the vector potential $\mathbf{A}$ is given by

$$\mathbf{J} = \frac{e}{m}(\mathbf{p} + e\mathbf{A})$$

$$
\begin{aligned}
&= \frac{e}{m}((p_x - \frac{eB}{2}y), (p_y + \frac{eB}{2}x), 0) \\
&= \frac{e}{2m\ell}(i({a_1}^\dagger - a_i) - (a_2 + {a_2}^\dagger), i({a_2}^\dagger - a_2) - (a_i + {a_1}^\dagger), 0)
\end{aligned}
$$

which may be represented by

$$
\mathbf{j} = j_0 \left(\begin{bmatrix} 0 & 0 & i \\ 0 & 0 & -1 \\ -i & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & -i & 0 \end{bmatrix}, 0 \right) \quad (j_0 \equiv \frac{e}{2m\ell}). \tag{46}
$$

Under the transformation Eqn. (40),

$$
\begin{aligned}
\mathbf{j} &\rightarrow T^\dagger \mathbf{j} T \\
&= j_0 \left(\begin{bmatrix} 0 & 0 & \sqrt{2}i \\ 0 & 0 & 0 \\ -\sqrt{2}i & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & -\mathcal{E}_1 \end{bmatrix}, 0 \right)
\end{aligned} \tag{47}
$$

This corresponds in Fock space to

$$
\mathbf{J} \rightarrow \frac{e}{2m\ell}(\sqrt{2}i({a_1}^\dagger - a_i), \sqrt{2}({a_1}^\dagger + a_i) - \mathcal{E}_1, 0) \tag{48}
$$

and shows immediately that the ground state current density (Hall current) is given by $-\frac{e}{2ml}\mathcal{E}_1 \equiv \frac{-eE_x}{B}$ in the y-direction. As is well-known, this treatment gives no longitudinal current (drift current) [2],[5] which arises, as is clear from Eqn.(48), from mixing of the Landau levels.

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References

[1] T. Chakraborty and P. Pietilainen, *The Quantum Hall Effects: Integral and Fractional.* 2 ed., Vol. 85. Solid-State Sciences, ed. M. Cardona et al. Berlin: Springer-Verlag, 1995.

[2] M. Stone,*The Quantum Hall Effect.* World Scientific Publishing Co. Pte. Ltd., 1992.

[3] A. Feldman and A.H. Kahn, Physical Review **B 1**,4584 (1970).

[4] Y. Dothan, M. Gell-Mann and Y. Ne'eman, Phys.Rev.Lett. **17**,148 (1965).

[5] S. J. Martinez and M. Stone, *Current Operators in the Lowest Landau Level*, preprint NSF-ITP-93-38, March 1993.